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Uniform Normal Approximation Orders for Families of Dominated Measures

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1. INTRODUCTION AND NOTATION

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $1 \leqslant s \leqslant \infty$. If $\mathbb{R}^k$ is endowed with the euclidean norm, denote by $\mathscr{L}_s(\Omega, \mathscr{A}, P, \mathbb{R}^k)$ the system of all $\mathscr{A}$-measurable $X\colon \Omega \to \mathbb{R}^k$ with $\|X\|_s < \infty$, where $\|X\|_s = (\int |X|^s \, dP)^{1/s}$ for $1 \leqslant s < \infty$ and $\|X\|_\infty = \inf\{c > 0\colon |X| \leqslant c \ P\text{-a.e.}\}$.

Let $X_n \in \mathscr{L}_2 \ (\Omega, \mathscr{A}, P, \mathbb{R}^k)$, $n \in \mathbb{N}$, be a sequence of independent and identically distributed (i.i.d.) random vectors with positive definite covariance matrix V. Put $S_n^* = (1/\sqrt{n}) \, V^{-1/2}(\sum_{\nu=1}^n (X_\nu - P[X_\nu]))$, where $P[X_\nu] = \int X_\nu \, dP$. Let $\mathscr{A}_n = \sigma(X_1, \ldots, X_n)$ be the σ-field generated by $X_1, \ldots, X_n$. If $\varphi \in \mathscr{L}_1$ $(\Omega, \mathscr{A}, P, \mathbb{R})$, let

$$d_1(\varphi, \mathscr{A}_n) := \inf\{\|\varphi - \psi\|_1 : \psi \ \mathscr{A}_n\text{-measurable}\},$$

the $\| \ \|_1$-distance of φ from the subspace $\mathscr{L}_1(\Omega, \mathscr{A}_n, P, \mathbb{R})$.

Let Φ be the distribution function of the standard normal distribution in $\mathbb{R}$. According to a well-known theorem of Renyi we have for each $\varphi \in \mathscr{L}_1$ $(\Omega, \mathscr{A}, P, \mathbb{R})$,

$$\sup_{t \in \mathbb{R}} |P[1_{\{S_n^* \leqslant t\}} \varphi] - \Phi(t) \, P[\varphi]|_{n \in \mathbb{N}} \to 0.$$

In this paper we investigate convergence rates of these expressions. In [4,

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640 45 2-2

Corollary 3], it was shown that, for i.i.d. $X_n \in \mathscr{L}_3(\Omega, \mathscr{A}, P, \mathbb{R})$ and $\varphi = 1_B$ with $B \in \mathscr{A}$, we have

$$\begin{aligned} d_1(\varphi, \mathscr{A}_n) = O(n^{-1/2}(\lg n)^\beta) \\ \Rightarrow \sup_{t \in \mathbb{R}} |P[1_{\{S_n^* \leqslant t\}} \varphi] - \Phi(t) P[\varphi]| &= O(n^{-1/2}); & \beta < -\tfrac{3}{2} \\ &= O(n^{-1/2} \lg \lg n); & \beta = -\tfrac{3}{2} \\ &= O(n^{-1/2}(\lg n)^{\beta + 3/2}); & \beta > -\tfrac{3}{2}, \end{aligned} \tag{I}$$

these convergence rates being optimal. It seems desirable to obtain the implication (I) for more general functions φ than indicator functions. If, e.g., φ is a density of a probability measure $Q | \mathscr{A}$ with respect to $P | \mathscr{A}$, implication (I) yields a convergence order for $\sup_{t \in \mathbb{R}} |Q(S_n^* \leqslant t) - \Phi(t)|$. Unfortunately implication (I) is not true any more for arbitrary densities φ: Example 1 shows that even if $d_1(\varphi, \mathscr{A}_n) = 0$ for all $n \in \mathbb{N}$ and X_n is standard normally distributed, implication (I) "extremely" fails. It turns out that we need suitable moment conditions for φ and X_n to guarantee implication (I). We prove that (I) holds if $\varphi \in \mathscr{L}_r(\mathbb{R})$ and $X_n \in \mathscr{L}_s(\mathbb{R})$ where $r = \infty$ if $s = 3$ and $r > 1 + 1/(s-3)$ if $s > 3$. Example 5 shows that these moment conditions are essentially optimal. We prove our result for $\mathbb{R}^k$-valued X_n and replace, moreover, $1_{\{S_n^* \leqslant t\}} = 1_{(-\infty, t]} \circ S_n^*$ by $f \circ S_n^*$ with Berry–Esseen functions $f: \mathbb{R}^k \to [-1, 1]$ (see Theorem 4). This result yields, e.g., convergence rates for

$$\sup_{Q \in \mathscr{Q}} \sup_{C \in \mathscr{C}} |Q(S_n^* \in C) - \Phi_{0,I}(C)|$$

where $\mathscr{Q}$ is a family of p-measures dominated by P, $\mathscr{C}$ is the class of all convex measurable sets of $\mathbb{R}^k$, and $\Phi_{0,I}$ is the standard normal distribution of $\mathbb{R}^k$ (see Corollary 6). Furthermore we prove a corresponding result (Theorem 7) using the $\| \ \|_r$-distance

$$d_r(\varphi, \mathscr{A}_n) := \inf\{ \|\varphi - \psi\|_r : \psi \ \mathscr{A}_n\text{-measurable} \}$$

instead of the $\| \ \|_1$-distance $d_1(\varphi, \mathscr{A}_n)$. Examples show that the convergence rates in this theorem as well as the moment conditions are optimal. We often write $P(S_n^* \leqslant t, \varphi)$ instead of $P[1_{\{S_n^* \leqslant t\}} \varphi]$ and $\Phi_{0,I}[f]$ instead of $\int f(x) \Phi_{0,I}(dx)$. Furthermore $F_n(x) = P\{S_n^* \leqslant x\}$, $x \in \mathbb{R}^k$, denotes the distribution function of S_n^*. If $X_1 \in \mathscr{L}_s(\Omega, \mathscr{A}, P, \mathbb{R}^k)$ has positive definite covariance matrix V, we write

$$\rho_s := P[|V^{-1/2}(X_1 - P[X_1])|^s].$$

If we write $c = c(., ., .)$ the parameters in the bracket are the only parameters the constant $(c > 0)$ depends upon.

In Section 2 we present our Results, in Section 3 we prove the Theorems of Section 2, and in Section 4 we prove the counterexamples of Section 2. Section 5 contains all auxiliary lemmata.

2. The Results

The following Example 1 shows that implication (I) does not hold for all $\varphi \in \mathscr{L}_1(\Omega, \mathscr{A}, P, \mathbb{R})$.

1. Example. Let $X_n, n \in \mathbb{N}$, be i.i.d. and standard normally distributed in $\mathbb{R}$. Then there exists $0 \leqslant \varphi \in \mathscr{L}_1$ such that

$$\text{(i)} \qquad d_1(\varphi, \mathscr{A}_n) = 0 \qquad \text{for all} \quad n \in \mathbb{N}$$

and

$$\text{(ii)} \qquad |P(S_n^* \leqslant 0, \varphi) - \Phi(0)\, P[\varphi]| \geqslant c \frac{1}{(\lg n)^2}, \qquad n \geqslant 3.$$

To formulate our results we need the following definition.

2. Definition. Let $X_n \in \mathscr{L}_3(\Omega, \mathscr{A}, P, \mathbb{R}^k), n \in \mathbb{N}$, be i.i.d. A function $f\colon \mathbb{R}^k \to [-1, 1]$ is a Berry–Esseen function iff f is Borel-measurable and

$$\left| \int f(ax+b)(F_n - \Phi_{0,I})(dx) \right| \leqslant \frac{c_f}{\sqrt{n}} \qquad \text{for} \quad 0 < a \leqslant 1, b \in \mathbb{R}^k,$$

where $c_f = c(f, P \circ X_1)$.

3. *Remark.* Let $X_n \in \mathscr{L}_3(\Omega, \mathscr{A}, P, \mathbb{R}^k), n \in \mathbb{N}$, be i.i.d. with positive definite covariance matrix.

(i) If $f\colon \mathbb{R}^k \to [-1, 1]$ is a Lipschitz function (i.e., $|f(x) - f(y)| \leqslant c|x-y|$), then f is a Berry–Esseen function with $c_f = c(k) \cdot c \cdot \rho_3$ (see [1, Theorem 17.8, p. 173]).

(ii) If $f := 1_C$, with $C \subset \mathbb{R}^k$ convex and Borel-measurable, then f is a Berry–Esseen function with $c_f = c(k) \cdot \rho_3$ (see [1, Corollary 17.2, p. 165]).

4. Theorem. *Let $X_n \in \mathscr{L}_s(\Omega, \mathscr{A}, P, \mathbb{R}^k)$, $n \in \mathbb{N}$, be i.i.d. with positive definite covariance matrix, where $3 \leqslant s < \infty$. Let $H \subset \mathscr{L}_r(\Omega, \mathscr{A}, P, \mathbb{R})$ with*

$\sup_{\varphi \in H} \|\varphi\|_r < \infty$. *Assume that* $r = \infty$ *if* $s = 3$ *and* $r > 1 + 1/(s-3)$ *if* $s > 3$. *Let* $\mathscr{F}$ *be a family of Berry–Esseen functions* $f: \mathbb{R}^k \to [-1, 1]$ *with* $\sup_{f \in \mathscr{F}} c_f < \infty$. *Then* $\sup_{\varphi \in H} d_1(\varphi, \mathscr{A}_n) = O(n^{-\alpha}(\lg n)^\beta)$ *implies*

$$\sup_{f \in \mathscr{F}, \varphi \in H} |P[(f \circ S_n^*)\varphi] - \Phi_{0,I}[f]\, P[\varphi]|$$

$$\begin{aligned} &= O(n^{-1/2}); && \alpha = \tfrac{1}{2}, \beta < -\tfrac{3}{2} \\ &= O(n^{-1/2} \lg \lg n); && \alpha = \tfrac{1}{2}, \beta = -\tfrac{3}{2} \\ &= O(n^{-1/2}(\lg n)^{\beta + 3/2}); && \alpha = \tfrac{1}{2}, \beta > -\tfrac{3}{2} \\ &= O(n^{-\alpha}(\lg n)^{\beta + \alpha}); && 0 < \alpha < \tfrac{1}{2}. \end{aligned}$$

The convergence rates of the preceding Theorem are optimal. This can be seen from Examples, given in [4], where even $H = \{1_B\}$ for some fixed $B \in \mathscr{A}$, $\mathscr{F} = \{1_{(-\infty, 0]}\}$, $k = 1$, and $X_n \in \mathscr{L}_\infty$. Example 5 will show that the moment assumptions on φ and X_n in Theorem 4 are essentially optimal.

A thorough examination of the proof of the d_1-inequality of Section 2 shows that if $r = 1 + 1/(s-3)$ $(s > 3)$ Theorem 4 also holds for the following cases: $0 < \alpha < \frac{1}{2}$ and $\beta \in \mathbb{R}$; $\alpha = \frac{1}{2}$ and $\beta < -s/2$; $\alpha = \frac{1}{2}$ and $\beta \geqslant -s/2 \cdot 1/(s-2)$.

We do not know whether it holds for the remaining case, i.e., $\alpha = \frac{1}{2}$ and $-s/2 \leqslant \beta < -s/2 \cdot 1/(s-2)$. The following example shows that for $r < 1 + 1/(s-3)$ $(s > 3)$ all four convergence orders given in Theorem 4 cannot be achieved any more. This example works with $k = 1$, $\mathscr{F} = \{1_{(-\infty, 0]}\}$, and $H = \{\varphi\}$.

5. Example. Let $s > 3$ and $r < 1 + 1/(s-3)$. There exist i.i.d. $X_n \in \mathscr{L}_s(\mathbb{R})$, $n \in \mathbb{N}$, a function $\varphi \in \mathscr{L}_r(\mathbb{R})$, and τ_1, τ_2 with $0 < \tau_1 < \frac{1}{2} < \tau_2$ such that

(i) $d_1(\varphi, \mathscr{A}_n) = O(n^{-\tau_2})$, and

(ii) $|P(S_n^* \leqslant 0, \varphi) - \Phi(0)\, P[\varphi]| \geqslant cn^{-\tau_1}$ for sufficiently large n.

This example shows that if $r < 1 + 1/(s-3)$ the convergence results of Theorem 4 are not true for each pair (α, β) with $\alpha = \frac{1}{2}$, $\beta \in \mathbb{R}$ and for each (α, β) with $\tau_1 < \alpha < \frac{1}{2}$, $\beta \in \mathbb{R}$.

6. Corollary. *Let* $X_n \in \mathscr{L}_s(\Omega, \mathscr{A}, P, \mathbb{R}^k)$, $n \in \mathbb{N}$, *be i.i.d. with positive definite covariance matrix where* $3 \leqslant s < \infty$. *Let* $\mathscr{Q} \ll P$ *be a family of p-measures with densities* φ_Q, $Q \in \mathscr{Q}$, *such that* $\sup_{Q \in \mathscr{Q}} \|\varphi_Q\|_r < \infty$. *Assume that* $r = \infty$ *if* $s = 3$ *and* $r > 1 + 1/(s-3)$ *if* $s > 3$. *Then* $\sup_{Q \in \mathscr{Q}} d_1(\varphi_Q, \mathscr{A}_n) = O(n^{-\alpha}(\lg n)^\beta)$ *implies*

$$\sup_{Q \in \mathscr{Q}, C \in \mathscr{C}} |Q(S_n^* \in C) - \Phi_{0,I}(C)|$$

$$\begin{aligned} &= O(n^{-1/2}); && \alpha = \tfrac{1}{2}, \beta < -\tfrac{3}{2} \\ &= O(n^{-1/2} \lg \lg n); && \alpha = \tfrac{1}{2}, \beta = -\tfrac{3}{2} \\ &= O(n^{-1/2} (\lg n)^{\beta + 3/2}); && \alpha = \tfrac{1}{2}, \beta > -\tfrac{3}{2} \\ &= O(n^{-\alpha} (\lg n)^{\beta + \alpha}); && 0 < \alpha < \tfrac{1}{2} \end{aligned}$$

where $\mathscr{C}$ is the system of all Borel-measurable convex subsets of $\mathbb{R}^k$.

Corollary 6 follows directly from Theorem 4 with $H = \{\varphi_Q : Q \in \mathscr{Q}\}$ and $\mathscr{F} = \{1_C : C \in \mathscr{C}\}$. Observe that $\mathscr{F}$ is a family of Berry–Esseen functions with $\sup\{c_f : f \in \mathscr{F}\} < \infty$ (see Remark 3).

Another application of Theorem 4 works with $H = \{\varphi_Q : Q \in \mathscr{Q}\}$ and $\mathscr{F} = \{f\}$, where f is a bounded Lipschitz function (see also Corollary 10).

In the following we use the $\| \ \|_r$-distance $d_r(\varphi, \mathscr{A}_n)$ instead of $d_1(\varphi, \mathscr{A}_n)$. Obviously $d_1(\varphi, \mathscr{A}_n) \leqslant d_r(\varphi, \mathscr{A}_n)$; hence the assumption $d_r(\varphi, \mathscr{A}_n) = O(n^{-\alpha}(\lg n)^{\beta})$ is stronger than the assumption $d_1(\varphi, \mathscr{A}_n) = O(n^{-\alpha}(\lg n)^{\beta})$.

If, however, $d_r(\varphi, \mathscr{A}_n) = O(d_1(\varphi, \mathscr{A}_n)) = O(n^{-\alpha}(\lg n)^{\beta})$, then the following Theorem yields better convergence rates under weaker moment conditions than Theorem 4.

7. THEOREM. *Let $X_n \in \mathscr{L}_s(\Omega, \mathscr{A}, P, \mathbb{R}^k)$, $n \in \mathbb{N}$, be i.i.d. with positive definite covariance matrix where $3 \leqslant s < \infty$. Let $H \subset \mathscr{L}_r(\Omega, \mathscr{A}, P, \mathbb{R})$ with $\sup_{\varphi \in H} \|\varphi\|_r < \infty$, where $r = 1 + 1/(s-1)$ (i.e., $1/r + 1/s = 1$). Let $\mathscr{F}$ be a family of Berry–Esseen functions $f: \mathbb{R}^k \to [-1, 1]$ with $\sup_{f \in \mathscr{F}} c_f < \infty$. Then $\sup_{\varphi \in H} d_r(\varphi, \mathscr{A}_n) = O(n^{-\alpha}(\lg n)^{\beta})$ implies*

$$\sup_{f \in \mathscr{F}, \varphi \in H} |P[(f \circ S_n^*)\varphi] - \Phi_{0,I}[f]\, P[\varphi]|$$

$$\begin{aligned} &= O(n^{-1/2}); && \alpha = \tfrac{1}{2}, \beta < -1 \\ &= O(n^{-1/2} \lg \lg n); && \alpha = \tfrac{1}{2}, \beta = -1 \\ &= O(n^{-1/2} (\lg n)^{\beta + 1}); && \alpha = \tfrac{1}{2}, \beta > -1 \\ &= O(n^{-\alpha} (\lg n)^{\beta}); && 0 < \alpha < \tfrac{1}{2}. \end{aligned}$$

The following Example shows that the convergence rates in Theorem 7 are optimal (even if $k = 1$, $H = \{\varphi\}$, and $\mathscr{F} = \{1_{(-\infty, 0]}\}$).

8. EXAMPLE. Let $X_n \in \mathscr{L}_3(\mathbb{R})$, $n \in \mathbb{N}$, be i.i.d. with positive variance and let $r \geqslant 1$. Assume that $P \circ X_1 = P \circ (-X_1)$ and that $P \circ X_1$ is nonatomic.

Then there exists a function $\varphi = \varphi_{\alpha,\beta} \in \mathscr{L}_r(\mathbb{R})$ such that

$$\text{(i)} \qquad d_r(\varphi, \mathscr{A}_n) = O(n^{-\alpha}(\lg n)^{\beta}),$$

and

$$\begin{aligned} \text{(ii)} \quad & |P(S_n^* \leqslant 0, \varphi) - \Phi(0)\, P[\varphi]| \\ & \geqslant c \cdot n^{-1/2} \lg \lg n; && \text{if} \quad \alpha = \tfrac{1}{2}, \beta = -1 \\ & \geqslant c \cdot n^{-1/2} (\lg n)^{\beta+1}; && \text{if} \quad \alpha = \tfrac{1}{2}, \beta > -1 \\ & \geqslant c \cdot n^{-\alpha} (\lg n)^{\beta}; && \text{if} \quad 0 < \alpha < \tfrac{1}{2} \end{aligned}$$

for sufficiently large n.

The next Example shows that the moment conditions on φ and X_n in Theorem 7 cannot be weakened.

9. EXAMPLE. Let $s \geqslant 3$ and $1 < r < 1 + 1/(s-1)$. Then there exist i.i.d. $X_n \in \mathscr{L}_s(\mathbb{R})$, $n \in \mathbb{N}$, a function $0 \leqslant \varphi \in \mathscr{L}_r(\mathbb{R})$, and τ with $0 < \tau < \frac{1}{2}$ such that

(i) $d_r(\varphi, \mathscr{A}_n) = 0$ for all $n \in \mathbb{N}$ and

(ii) $|P(S_n^* \leqslant 0, \varphi) - \Phi(0)\, P[\varphi]| \geqslant cn^{-\tau}$ for sufficiently large $n \in \mathbb{N}$.

10. COROLLARY. *Let $X_n \in \mathscr{L}_s(\Omega, \mathscr{A}, P, \mathbb{R}^k)$, $n \in \mathbb{N}$, be i.i.d. with positive definite covariance matrix where $3 \leqslant s < \infty$. Let $\mathscr{Q} \ll P$ be a family of p-measures with densities φ_Q, $Q \in \mathscr{Q}$, such that $\sup_{Q \in \mathscr{Q}} \|\varphi_Q\|_r < \infty$, where $1/r + 1/s = 1$.*

Then $\sup_{Q \in \mathscr{Q}} d_r(\varphi_Q, \mathscr{A}_n) = O(n^{-\alpha}(\lg n)^{\beta})$ implies that for each Lipschitz function $f\colon \mathbb{R}^k \to [-1, 1]$

$$\begin{aligned} \sup_{Q \in \mathscr{Q}} |Q[f \circ S_n^*] - \Phi_{0,I}[f]| &= O(n^{-1/2}); && \alpha = \tfrac{1}{2}, \beta < -1 \\ &= O(n^{-1/2} \lg \lg n); && \alpha = \tfrac{1}{2}, \beta = -1 \\ &= O(n^{-1/2} (\lg n)^{\beta+1}); && \alpha = \tfrac{1}{2}, \beta > -1 \\ &= O(n^{-\alpha} (\lg n)^{\beta}); && 0 < \alpha < \tfrac{1}{2}. \end{aligned}$$

Corollary 10 follows directly from Theorem 8 with $H = \{\varphi_Q : Q \in \mathscr{Q}\}$ and $\mathscr{F} = \{f\}$. Observe that a Lipschitz function is a Berry–Esseen function (see Remark 3).

Another application of Theorem 8 works with $H = \{\varphi_Q : Q \in \mathscr{Q}\}$ and $\mathscr{F} = \{1_C : C \subset \mathbb{R}^k$ convex and measurable$\}$ (see also Corollary 6).

3. Proof of the Theorems

In this section we prove two inequalities which directly imply our main results of Section 2, namely, Theorem 4 and Theorem 7.

(A) d_1-Inequality. *Let $X_n \in \mathscr{L}_s(\mathbb{R}^k)$, $n \in \mathbb{N}$, be i.i.d. with positive definite covariance matrix V, where $3 \leqslant s < \infty$. Let $\varphi \in \mathscr{L}_r(\mathbb{R})$ with $r = \infty$ if $s = 3$ and $r > 1 + 1/(s-3)$ if $s > 3$. Let $f: \mathbb{R}^k \to [-1, 1]$ be a Berry–Esseen function. Then there exists a constant $c = c(r, s, k)$ such that for all $j \leqslant n/2$*

$$|P[(f \circ S_n^*)\varphi] - \Phi_{0,I}[f]\, P[\varphi]| \leqslant \frac{c\rho_s + 4c_f}{\sqrt{n}} \left(\|\varphi\|_r + \sum_{v=2}^{j} \sqrt{\frac{\lg v}{v}}\, d_1(\varphi, \mathscr{A}_v) \right) + 2\, d_1(\varphi, \mathscr{A}_j)$$

where c_f is the constant occurring in the definition of a Berry–Esseen function.

Proof. W.l.g. we may assume that $P[X_1] = 0$ and $V = I$; otherwise consider $V^{-1/2}(X_n - P[X_n])$, $n \in \mathbb{N}$.

Put $\varepsilon_v := d_1(\varphi, \mathscr{A}_v) = \inf\{\|\varphi - \psi\|_1 : \psi\ \ \mathscr{A}_v\text{-measurable}\}$. According to Shintani and Ando [5] there exist $\mathscr{A}_v$-measurable functions $\varphi_v : \Omega \to \mathbb{R}$ with

$$\|\varphi - \varphi_v\|_1 = \varepsilon_v, \qquad v \in \mathbb{N}. \tag{1}$$

Now let j and n with $j \leqslant n/2$ be fixed. Put

$$m(0) = 0, \qquad \varepsilon_0 = \|\varphi\|_1. \tag{2}$$

If $m(i) < j$ is defined let

$$m(i+1) = j, \qquad \text{if} \quad \varepsilon_v \geqslant \tfrac{1}{4}\varepsilon_{m(i)} \text{ for } m(i) < v \leqslant j \tag{3}$$

otherwise let

$$m(i+1) = \min\{v \in \mathbb{N}: m(i) < v \leqslant j,\ \varepsilon_v < \tfrac{1}{4}\varepsilon_{m(i)}\}. \tag{4}$$

According to the inductive definition of $m(i)$ given in (2)–(4) we obtain $l \in \mathbb{N} \cup \{0\}$ and $0 = m(0) < m(1) < \cdots < m(l) < m(l+1) = j$ with

$$\varepsilon_{m(i)} < \tfrac{1}{4}\varepsilon_{m(i-1)}, \qquad 1 \leqslant i \leqslant l \tag{5}$$

$$\varepsilon_{m(i)} \leqslant 4\varepsilon_v, \qquad 0 \leqslant i \leqslant l,\ m(i) \leqslant v < m(i+1). \tag{6}$$

By (5) and (2) we have

$$\varepsilon_{m(i)} \leqslant (1/4^i)\, \|\varphi\|_1, \qquad 0 \leqslant i \leqslant l. \tag{7}$$

Put

$$\psi_{m(i)} = \varphi_{m(i)} - \varphi_{m(i-1)}, \qquad 1 \leqslant i \leqslant l+1, \text{ where } \varphi_{m(0)} = 0. \tag{8}$$

By (1) and (2) we have

$$P[|\psi_{m(i)}|] \leqslant 2\varepsilon_{m(i-1)}, \qquad 1 \leqslant i \leqslant l+1. \tag{9}$$

Let $L(\psi)$ be the left side of the asserted formula, i.e.,

$$L(\psi) := |P[(f \circ S_n^*)\,\psi] - \Phi_{0,I}[f]\; P[\psi]|.$$

By (8) we have $\varphi = \varphi - \varphi_j + \sum_{i=1}^{l+1} \psi_{m(i)}$.

Since $|f| \leqslant 1$ this implies according to (1)

$$L(\varphi) \leqslant 2\varepsilon_j + \sum_{i=1}^{l+1} L(\psi_{m(i)}). \tag{10}$$

In the following let c_v denote constants only depending on r, s, and k.

Since f is a Berry–Esseen function, we can apply Lemma 2 for each $v = m(i)$. As $1 \leqslant m(i) \leqslant j \leqslant n/2$ for $i = 1,..., l+1$ there consequently exists a constant c_1 such that

$$\begin{aligned} &|P(f \circ S_n^* \mid \mathscr{A}_{m(i)}) - \Phi_{0,I}[f]| \\ &\quad \leqslant \sqrt{2}\,\frac{c_f}{\sqrt{n}} + c_1\left(\frac{m(i)}{n} + \sqrt{\frac{m(i)}{n}}\,|S_{m(i)}^*|\right). \end{aligned} \tag{11}$$

As $\psi_{m(i)}$ is $\mathscr{A}_{m(i)}$-measurable, we obtain by (11) for $1 \leqslant i \leqslant l+1$

$$\begin{aligned} L(\psi_{m(i)}) &= |P[(P(f \circ S_n^* \mid \mathscr{A}_{m(i)}) - \Phi_{0,I}[f])\,\psi_{m(i)}]| \\ &\leqslant \left(\sqrt{2}\,\frac{c_f}{\sqrt{n}} + c_1\sqrt{\frac{m(i)}{n}}\right) P[|\psi_{m(i)}|] \\ &\quad + c_1\sqrt{\frac{m(i)}{n}}\, P[|\psi_{m(i)} S_{m(i)}^*|]. \end{aligned} \tag{12}$$

Put $A_v := \{|S_v^*| > \rho_s^{1/s}\sqrt{(s-1)\,k \lg v}\}$. For $1 \leqslant i \leqslant l+1$ we have

$$\begin{aligned} &P[|\psi_{m(i)} S_{m(i)}^*|] \\ &\quad \leqslant \rho_s^{1/s}\sqrt{(s-1)\,k \lg m(i)}\; P[|\psi_{m(i)}|] + \int_{A_{m(i)}} |\psi_{m(i)} S_{m(i)}^*|\; dP. \end{aligned}$$

Hence we obtain from (12) for $1 \leqslant i \leqslant l+1$

$$L(\psi_{m(i)}) \leqslant \sqrt{2}\, \frac{c_f}{\sqrt{n}} P[|\psi_{m(i)}|] + 2c_1 \sqrt{(s-1)\,k}\, \frac{\rho_s^{1/s}}{\sqrt{n}} \sqrt{m(i) \lg(m(i)+2)}\, P[|\psi_{m(i)}|] + c_1 \frac{1}{\sqrt{n}} \sqrt{m(i)} \int_{A_{m(i)}} |\psi_{m(i)} S^*_{m(i)}|\, dP. \tag{13}$$

Now we prove the three relations

$$\sum_{i=1}^{l+1} P[|\psi_{m(i)}|] \leqslant \frac{8}{3} \|\varphi\|_1 \tag{14}$$

$$\sum_{i=1}^{l+1} \sqrt{m(i) \lg[m(i)+2]}\, P[|\psi_{m(i)}|] \leqslant c_2 \left(\|\varphi\|_1 + \sum_{\nu=1}^{j} \sqrt{\frac{\lg(\nu+2)}{\nu}}\, \varepsilon_\nu \right) \tag{15}$$

$$\sum_{i=1}^{l+1} \sqrt{m(i)} \int_{A_{m(i)}} |\psi_{m(i)} S^*_{m(i)}|\, dP \leqslant c_3 \rho_s \|\varphi\|_r. \tag{16}$$

From (10) and (13)–(16) we obtain the assertion as

$$\sqrt{2}\, c_f \tfrac{8}{3} \|\varphi\|_1 \leqslant 4 c_f \|\varphi\|_r$$

and

$$\begin{aligned}
\rho_s^{1/s} \cdot \left(\|\varphi\|_1 + \sum_{\nu=1}^{j} \sqrt{\frac{\lg(\nu+2)}{\nu}}\, \varepsilon_\nu \right)
&\leqslant \rho_s^{1/s} \left(\|\varphi\|_1 + \sqrt{\lg 3}\, \varepsilon_1 + 2 \sum_{\nu=2}^{j} \sqrt{\frac{\lg \nu}{\nu}}\, \varepsilon_\nu \right) \\
&\leqslant (1+\sqrt{\lg 3})\, \rho_s^{1/s} \|\varphi\|_1 + 2\rho_s^{1/s} \sum_{\nu=2}^{j} \sqrt{\frac{\lg \nu}{\nu}}\, \varepsilon_\nu \\
&\leqslant (1+\sqrt{\lg 3})\, \rho_s \left(\|\varphi\|_r + \sum_{\nu=2}^{j} \sqrt{\frac{\lg \nu}{\nu}}\, \varepsilon_\nu \right),
\end{aligned}$$

where the last inequality follows from $\|\varphi\|_1 \leqslant \|\varphi\|_r$ and $\rho_s \geqslant 1$.

Ad (14). We have by (9) and (7)

$$\sum_{i=1}^{l+1} P[|\psi_{m(i)}|] \leqslant 2 \sum_{i=1}^{l+1} \varepsilon_{m(i-1)} = 2 \sum_{i=0}^{l} \varepsilon_{m(i)} \leqslant 2 \|\varphi\|_1 \sum_{i=0}^{l} \frac{1}{4^i} \leqslant \frac{8}{3} \|\varphi\|_1.$$

Ad (15). Put $a_\mu = \sqrt{\mu \lg(\mu+2)}$, $x_{m(i)} = P[|\psi_{m(i)}|]$, $1 \leqslant i \leqslant l+1$, and $x_\mu = 0$ elsewhere. Using that a_μ/μ is decreasing, we have $a_\mu \leqslant \sum_{\nu=1}^{\mu} (a_\nu/\nu)$ and hence

$$\begin{aligned}\sum_{i=1}^{l+1} \sqrt{m(i) \lg[m(i)+2]}\, P[|\psi_{m(i)}|] \\ &= \sum_{\mu=1}^{j} x_\mu a_\mu \leqslant \sum_{\mu=1}^{j} x_\mu \sum_{\nu=1}^{\mu} \frac{a_\nu}{\nu} \\ &= \sum_{\nu=1}^{j} \frac{a_\nu}{\nu} \sum_{\mu=\nu}^{j} x_\mu = \sum_{\nu=1}^{j} \sqrt{\frac{\lg(\nu+2)}{\nu}} \sum_{\mu=\nu}^{j} x_\mu. \qquad (17)\end{aligned}$$

If $m(i-1) < \nu \leqslant m(i)$ and $1 \leqslant i \leqslant l+1$, we have according to (9) and (5)

$$\begin{aligned}\sum_{\mu=\nu}^{j} x_\mu &= x_{m(i)} + \cdots + x_{m(l+1)} = P[|\psi_{m(i)}|] + \cdots + P[|\psi_{m(l+1)}|] \\ &\leqslant 2\varepsilon_{m(i-1)} + \cdots + 2\varepsilon_{m(l)} \\ &\leqslant 2\varepsilon_{m(i-1)} \left[1 + \sum_{\xi=1}^{\infty} \frac{1}{4^\xi}\right] \leqslant \frac{8}{3} \varepsilon_{m(i-1)}. \qquad (18)\end{aligned}$$

Hence we have

if $m(i-1) < \nu < m(i)$ and $1 \leqslant i \leqslant l+1$, then by (18) and (6)

$$\sum_{\mu=\nu}^{j} x_\mu \leqslant \frac{8}{3} \varepsilon_{m(i-1)} \leqslant \frac{8}{3} 4\varepsilon_\nu; \qquad (19)$$

if $\nu = m(i)$ and $2 \leqslant i \leqslant l+1$, then by (18)

$$\begin{aligned}\sqrt{\frac{\lg(\nu+2)}{\nu}} \sum_{\mu=\nu}^{j} x_\mu &\leqslant \sqrt{\frac{\lg(\nu+2)}{\nu}} \frac{8}{3} \varepsilon_{m(i-1)} \\ &\leqslant \frac{8}{3} \sqrt{\frac{\lg[m(i-1)+2]}{m(i-1)}} \varepsilon_{m(i-1)}; \qquad (20)\end{aligned}$$

if $\nu = m(1)$, then by (18) and (2)

$$\begin{aligned}\sqrt{\frac{\lg(\nu+2)}{\nu}} \sum_{\mu=\nu}^{j} x_\mu &\leqslant \sqrt{\frac{\lg(\nu+2)}{\nu}} \cdot \frac{8}{3} \varepsilon_0 \\ &= \frac{8}{3} \sqrt{\frac{\lg(m(1)+2)}{m(1)}} \|\varphi\|_1 \leqslant c_4 \|\varphi\|_1. \qquad (21)\end{aligned}$$

Now (17), (19), (20), and (21) imply (15). Therefore it remains to prove (16). We prove (16) at first for the case $s > 3$ and hence $r < \infty$.

Ad (16). Let r' fulfill $1/r' + 1/r = 1$ and s' fulfill $1/s' + 1/s = 1$. As $r > (s-2)/(s-3)$ we have

$$s > 2 + \frac{r}{r-1}; \qquad 1 < s' < r; r' < s-2. \tag{22}$$

According to (22) there exists $\alpha \in (0, 1)$ with

$$s' = \alpha \cdot 1 + (1-\alpha) r \quad \text{and hence} \quad \alpha = \frac{1}{r-1}(r-s') \in (0, 1). \tag{23}$$

Let $1 < a < (4^{\alpha/s'})^2$, then $\sqrt{a}/4^{\alpha/s'} < 1$.

Now put

$$M_0 = \{1 \leqslant i \leqslant l+1 : m(i) \leqslant a^i\}$$
$$M_1 = \{1 \leqslant i \leqslant l+1 : m(i) > a^i\}.$$

We prove that

$$D := \sum_{i \in M_0} \sqrt{m(i)} \int_{A_{m(i)}} |\psi_{m(i)} S^*_{m(i)}| \, dP \leqslant c_5 \rho_s \|\varphi\|_r \tag{16$_1$}$$

$$E := \sum_{i \in M_1} \sqrt{m(i)} \int_{A_{m(i)}} |\psi_{m(i)} S^*_{m(i)}| \, dP \leqslant c_6 \rho_s \|\varphi\|_r. \tag{16$_2$}$$

Obviously $(16)_1$ and $(16)_2$ imply (16).

Ad $(16)_1$. We have by Hölder and Lemma 7 using the definition of M_0

$$D \leqslant \sum_{i \in M_0} \sqrt{m(i)} \|S^*_{m(i)}\|_s \|\psi_{m(i)}\|_{s'} \leqslant c_7 \rho_s^{1/s} \sum_{i \in M_0} (\sqrt{a})^i \|\psi_{m(i)}\|_{s'}. \tag{24}$$

As $1/\alpha > 1$ and $(1/\alpha)' = (1/\alpha)/(1/\alpha - 1) = 1/(1-\alpha)$, we have according to Hölder's inequality and (23)

$$P[|\psi_{m(i)}|^{s'}] = P[|\psi_{m(i)}|^{\alpha} |\psi_{m(i)}|^{(1-\alpha)r}] \leqslant P[|\psi_{m(i)}|]^{\alpha} \, P[|\psi_{m(i)}|^r]^{1-\alpha}.$$

Using (9) and (7) we obtain

$$\|\psi_{m(i)}\|_{s'} \leqslant \left(\frac{2}{4^{i-1}} \|\varphi\|_1\right)^{\alpha/s'} P[|\psi_{m(i)}|^r]^{(1-\alpha)/s'}. \tag{25}$$

By (1) and Lemma 5, we have $\|\varphi_v\|_r \leqslant 2\|\varphi\|_r$; hence (8) implies

$$\|\psi_{m(i)}\|_r \leqslant 4\|\varphi\|_r. \tag{26}$$

From (25), (26), and (23) we obtain

$$\|\psi_{m(i)}\|_{s'} \leqslant 8^{\alpha/s'} \|\varphi\|_1^{\alpha/s'} \frac{1}{(4^{\alpha/s'})^i} (4\|\varphi\|_r)^{r(1-\alpha)/s'}$$

$$\leqslant c_8 \|\varphi\|_r \left(\frac{1}{4^{\alpha/s'}}\right)^i. \tag{27}$$

From (24) and (27) we obtain

$$D \leqslant c_9 \rho_s^{1/s} \|\varphi\|_r \sum_{i \in M_0} \left(\frac{\sqrt{a}}{4^{\alpha/s'}}\right)^i \leqslant c_5 \rho_s \|\varphi\|_r. \tag{28}$$

Hence we have proved $(16)_1$.

Ad $(16)_2$. Using the Hölder inequality we obtain from (26)

$$E \leqslant \sum_{i \in M_1} \sqrt{m(i)}\, \|S^*_{m(i)} 1_{A_{m(i)}}\|_{r'} \|\psi_{m(i)}\|_r$$

$$\leqslant 4\|\varphi\|_r \sum_{i \in M_1} \sqrt{m(i)}\, \|S^*_{m(i)} 1_{A_{m(i)}}\|_{r'}. \tag{29}$$

We have for $m \geqslant 2$—as $\int |Y|\, dP \leqslant \sum_{v=0}^{\infty} P\{|Y| > v\}$—

$$\|S^*_m 1_{A_m}\|_{r'}^{r'}$$

$$\leqslant \int |S^*_m|^{r'} 1_{\{|S^*_m| > \sqrt{(s-1)\lg m}\}}\, dP$$

$$= [(s-1)\lg m]^{r'/2} \int \left|\frac{S^*_m}{\sqrt{(s-1)\lg m}}\right|^{r'} 1_{\{|S^*_m|/\sqrt{(s-1)\lg m} > 1\}}\, dP$$

$$\leqslant 2(s-1)^{r'/2} (\lg m)^{r'/2} \sum_{v=1}^{\infty} P\left\{\left|\frac{S^*_m}{\sqrt{(s-1)\lg m}}\right|^{r'} > v\right\}$$

$$= 2(s-1)^{r'/2} (\lg m)^{r'/2} \sum_{v \in \mathbb{N}} P\{|S^*_m| > v^{1/r'}(s-1)^{1/2} \sqrt{\lg m}\}$$

and hence according to Lemma 6

$$\leqslant c_{10} (\lg m)^{r'/2} \rho_s \sum_{v \in \mathbb{N}} \frac{1}{v^{s/r'}(s-1)^{s/2} (\lg m)^{s/2}} \frac{1}{m^{(s-2)/2}}.$$

Therefore

$$\|S^*_m 1_{A_m}\|_{r'}^{r'} \leqslant c_{11} \cdot \rho_s \frac{1}{m^{(s-2)/2}} \frac{1}{(\lg m)^{(s-r')/2}}$$

and hence

$$\sqrt{m(i)}\,\|S^*_{m(i)}1_{A_{m(i)}}\|_{r'}$$

$$\leqslant c_{12}\rho_s^{1/r'}\frac{1}{m(i)^{((s-2)/2r')-1/2}}\frac{1}{(\lg m(i))^{(s-r')/2r'}}. \tag{30}$$

Let $\delta=\delta(r,s):=(s-2)/2r'-\frac{1}{2}$. From (29), (30), and $m(i)\geqslant a^i$ we obtain

$$E\leqslant c_{13}\|\varphi\|_r\rho_s\sum_{i\in M_1}\frac{1}{(a^\delta)^i}\frac{1}{i^{(s-r')/2r'}}. \tag{31}$$

As $\delta>0$ (here we use for the first time $r>1+1/(s-3)$) and $a>1$, (31) implies $(16)_2$. Thus the result is proven for the case $r<\infty$.

It remains to prove formula (16) for $r=\infty$, $s=3$. Therefore, it suffices to prove $(16)_1$ and $(16)_2$ with

$$M_0=\{1\leqslant i\leqslant l+1: m(i)\leqslant a^i\},\qquad M_1=\{1\leqslant i\leqslant l+1: m(i)>a^i\}$$

where $1<a<4^{2/3}$. Since $(16)_1$ follows by similar methods as for the case $r<\infty$ it remains to prove $(16)_2$. Since

$$\int_{A_{m(i)}}|\psi_{m(i)}S^*_{m(i)}|\,dP\leqslant 2\int_{A_{m(i)}}|S^*_{m(i)}|\,dP\,\|\varphi\|_\infty$$

we have to prove

$$\sum_{i\in M_1}\sqrt{m(i)}\int_{A_{m(i)}}|S^*_{m(i)}|\,dP\leqslant c_6\rho_3. \tag{32}$$

For the dimension $k=1$ relation (32) was proven in [4, proof of Theorem 2, formula (15)]. Let $X_n:=(X_{n,1},\dots,X_{n,k})$, and $S^*_{m,v}:=(1/\sqrt{m})\sum_{n=1}^m X_{n,v}$ for $v=1,\dots,k$. Since $V=I$, we have $\sigma(X_{n,v})=1$ and $\rho_{3,v}=P[|X_{1,v}|^3]\leqslant\rho_3$. Consequently we have for $v=1,\dots,k$

$$\sum_{i\in M_1}\sqrt{m(i)}\int|S^*_{m(i),v}|\,1_{\{|S^*_{m(i),v}|>\rho_{3,v}^{1/3}\sqrt{2\lg m(i)}\}}\,dP\leqslant c_{14}\rho_{3,v}.$$

Hence (32) follows from

$$|S^*_{m(i)}|\,1_{A_{m(i)}}\leqslant\sqrt{k}\sum_{v=1}^k|S^*_{m(i),v}|\,1_{\{|S^*_{m(i),v}|>\rho_{3,v}^{1/3}\sqrt{2\lg m(i)}\}}$$

using $\rho_{3,v}\leqslant\rho_3$.

This d_1-Inequality (A) directly implies Theorem 4: Apply (A) to $j=j(n)=[n/\lg n]$.

The following d_r-Inequality implies Theorem 7: Put $j=j(n)=[n/2]$.

(B) d_r-INEQUALITY. *Let $X_n \in \mathscr{L}_s(\mathbb{R}^k)$, $n \in \mathbb{N}$, be i.i.d. with positive definite covariance matrix V, where $3 \leqslant s < \infty$; and let $\varphi \in \mathscr{L}_r(\mathbb{R})$ where $1/s + 1/r = 1$. Let $f\colon \mathbb{R}^k \to [-1, 1]$ be a Berry–Esseen function. Then there exists a constant $c = c(s, k)$ such that for all $j \leqslant n/2$*

$$|P[(f \circ S_n^*)\,\varphi] - \Phi_{0,I}[f]\,P[\varphi]|$$
$$\leqslant \frac{c\rho_s + 4c_f}{\sqrt{n}}\left(\|\varphi\|_r + \sum_{v=2}^{j} \frac{1}{\sqrt{v}}\, d_r(\varphi, \mathscr{A}_v)\right) + 2\, d_r(\varphi, \mathscr{A}_j)$$

where c_f is the constant occurring in the definition of a Berry–Esseen function.

Proof. The proof runs similarly as the proof of the d_1-Inequality (A). Let $P[X_1] = 0$, $V = I$.

There exist $\mathscr{A}_v$-measurable $\varphi_v : \Omega \to \mathbb{R}$ with

$$\|\varphi - \varphi_v\|_r = d_r(\varphi, \mathscr{A}_v) =: \varepsilon_v, \qquad v \in \mathbb{N}. \tag{1}$$

Let j and n with $j \leqslant n/2$ be fixed. Put

$$m(0) := 0, \qquad \varepsilon_0 := \|\varphi\|_r. \tag{2}$$

Define $m(i)$ as in (A). Then (5)–(7) of (A) hold with $\|\varphi\|_r$ instead of $\|\varphi\|_1$. Define $\psi_{m(i)}$ and $L(\psi)$ as in (A). Then (9)–(12) hold, too. To prove the assertion it suffices to prove

$$\sum_{i=1}^{l+1} P[|\psi_{m(i)}|] \leqslant \frac{8}{3}\,\|\varphi\|_r \tag{14$'$}$$

$$\sum_{i=1}^{l+1} \sqrt{m(i)}\, P[|\psi_{m(i)}|] \leqslant \frac{40}{3}\left(\|\varphi\|_r + \sum_{v=1}^{j} \frac{\varepsilon_v}{\sqrt{v}}\right) \tag{15$'$}$$

$$\sum_{i=1}^{l+1} \sqrt{m(i)}\, P[|\psi_{m(i)} S^*_{m(i)}|] \leqslant c\rho_s\left(\|\varphi\|_r + \sum_{v=1}^{j} \frac{\varepsilon_v}{\sqrt{v}}\right). \tag{16$'$}$$

The proof of (14)′ runs as the proof of (14) in (A). To show (15)′ it suffices to prove

$$\sum_{i=1}^{l+1} \sqrt{m(i)}\, \|\psi_{m(i)}\|_r \leqslant \frac{40}{3}\left(\|\varphi\|_r + \sum_{v=1}^{j} \frac{\varepsilon_v}{\sqrt{v}}\right). \tag{15$''$}$$

The proof of (15)″ runs as the proof of (15) in (A), if we put $a_\mu = \sqrt{\mu}$, $x_{m(i)} = \|\psi_{m(i)}\|_r$.

Furthermore we obtain using the Hölder inequality and Lemma 7

$$\sum_{i=1}^{l+1} \sqrt{m(i)}\, P[|\psi_{m(i)} S^*_{m(i)}|] \leqslant \sum_{i=1}^{l+1} \sqrt{m(i)}\, \|\psi_{m(i)}\|_r \|S^*_{m(i)}\|_s$$
$$\leqslant c\rho_s \sum_{i=1}^{l+1} \sqrt{m(i)}\, \|\psi_{m(i)}\|_r.$$

Hence (16)′ follows from (15)″.

4. Proof of the Examples

In this section we give the proofs of the five examples of Section 2.

Proof of Example 1. Let $g(t) = (e^{t^2/2}/t(\lg t)^2)\, 1_{[2,\infty)}(t)$, $t \in \mathbb{R}$, and put $\varphi = g \circ X_1$. Then $0 \leqslant \varphi \in \mathscr{L}_1(\Omega, \mathscr{A}, P, \mathbb{R})$ and $d_1(\varphi, \mathscr{A}_n) = 0$ for all $n \in \mathbb{N}$. It remains to prove (ii). Using Lemma 1 we obtain for $n \geqslant 3$

$$\begin{aligned}
&|P(S_n^* \leqslant 0, \varphi) - \Phi(0)\, P[\varphi]| \\
&= \left| \int g \circ X_1\, P(S_n^* \leqslant 0 \,|\, \mathscr{A}_1)\, dP - \int \Phi(0)\, g \circ X_1\, dP \right| \\
&= \left| \int g \circ X_1 \left(\Phi\left(-\frac{1}{\sqrt{n-1}} X_1 \right) - \Phi(0) \right) dP \right| \\
&= \int_2^\infty g(t) \left(\Phi(0) - \Phi\left(-\frac{1}{\sqrt{n-1}}\, t \right) \right) P \circ X_1(dt) \\
&= \frac{1}{\sqrt{2\pi}} \int_2^\infty \frac{1}{t(\lg t)^2} \left(\Phi(0) - \Phi\left(-\frac{1}{\sqrt{n-1}}\, t \right) \right) dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{2/\sqrt{n-1}}^\infty \frac{1}{u(\lg u\sqrt{n-1})^2} (\Phi(0) - \Phi(-u))\, du \\
&\geqslant c_1 \int_2^3 \frac{1}{u(\lg 3\sqrt{n-1})^2}\, du \geqslant c\, \frac{1}{(\lg n)^2}.
\end{aligned}$$

Proof of Example 5. There exist i.i.d. nonatomic X_n, $n \in \mathbb{N}$, with variance 1, such that $P \circ X_1 = P \circ (-X_1)$ and $P\{X_1 > t\} \sim 1/t^s(\lg t)^2$ for $t \to \infty$. Then $X_n \in \mathscr{L}_s(\mathbb{R})$ and $P[X_n] = 0$. As $r < 1 + 1/(s-3)$ we have $s < 2 + r/(r-1)$ and hence there exists δ with

$$\tfrac{1}{2} < \delta < 1 \qquad \text{and} \qquad \delta(s - r/(r-1)) < 1. \tag{1}$$

By (1) there exists τ_2 with

$$\tfrac{1}{2} < \tau_2 < \delta \tag{2}$$

$$s\,\delta(1-r) + (\tau_2 + 1)\, r > 1. \tag{3}$$

Then by (2)

$$\tau_1 := \tau_2/2\delta < \tfrac{1}{2}. \tag{4}$$

Let $\varphi_\nu := (\lg \nu)^2 \, \nu^{s\delta - (\tau_2+1)} 1_{\{X_\nu > \nu^\delta\}}$ and put $\varphi = \sum_{\nu \in \mathbb{N}} \varphi_\nu$. At first we show that

$$\varphi \in \mathscr{L}_r(\mathbb{R}). \tag{5}$$

Since $\varphi_\nu \geqslant 0$, $\nu \in \mathbb{N}$, are independent and $s\delta - (\tau_2 + 1) \geqslant 0$, according to [2, Lemma 1, p. 358], relation (5) is shown if we prove

$$\sum_{\nu \in \mathbb{N}} P[\varphi_\nu^r] < \infty. \tag{6}$$

As

$$P[\varphi_\nu^r] = (\lg \nu)^{2r} \frac{1}{\nu^{(\tau_2+1-s\delta)r}} P\{X_\nu > \nu^\delta\}$$

$$\leqslant c_1 (\lg \nu)^{2r-2} \frac{1}{\nu^{s\delta(1-r)+(\tau_2+1)r}},$$

relation (3) implies (6).

Furthermore we have

$$d_1(\varphi, \mathscr{A}_n) \leqslant \sum_{\nu > n} P[\varphi_\nu] \leqslant c_1 \sum_{\nu > n} \frac{1}{\nu^{\tau_2+1}} \leqslant c_2 n^{-\tau_2}$$

i.e., (i) holds. It remains to prove (ii). As $P \circ X_1 = P \circ (-X_1)$ and $P \circ X_1$ is nonatomic Lemma 8 yields

$$\Phi(0)\, P[\varphi_\nu] - P(S_n^* \leqslant 0, \varphi_\nu) \geqslant 0 \qquad \text{for} \quad \nu, n \in \mathbb{N}. \tag{7}$$

Now we show that for some $\nu_0 \in \mathbb{N}$ there holds

$$\Phi(0)\, P[\varphi_\nu] - P(S_n^* \leqslant 0, \varphi_\nu)$$

$$\geqslant c_3 \frac{1}{\sqrt{n}} \frac{1}{\nu^{\tau_2+1-\delta}} \qquad \text{for} \quad \nu_0 \leqslant \nu \leqslant n^{1/2\delta}. \tag{8}$$

To prove (8) we apply Lemma 3 with $k=1$, $a = \nu^\delta$, and $B = \{S_1^* \geqslant a\} = \{X_1 \geqslant a\}$ and we obtain for all ν with $c(P \circ X_1)^{1/\delta} \leqslant \nu \leqslant n^{1/2\delta}$

$$\Phi(0)\, P[\varphi_\nu] - P(S_n^* \leqslant 0, \varphi_\nu)$$

$$= (\lg \nu)^2 \, \nu^{s\delta - (\tau_2+1)} (\Phi(0)\, P\{X_1 > \nu^\delta\} - P\{S_n^* \leqslant 0, X_1 > \nu^\delta\})$$

$$\geqslant c (\lg \nu)^2 \, \nu^{s\delta - (\tau_2+1)} \frac{1}{\sqrt{n}} \nu^\delta P\{X_1 > \nu^\delta\}.$$

Since $P\{X_1 > t\} \sim 1/t^s(\lg t)^2$ this implies for $v_0 \leqslant v \leqslant n^{1/2\delta}$ with appropriate $v_0 \in \mathbb{N}$

$$\geqslant c_3 \frac{1}{\sqrt{n}} \frac{1}{v^{\tau_2+1-\delta}}$$

i.e., (8) is shown.

As $0 < \tau_2 + 1 - \delta < 1$ by (1), (2) we obtain from (7) and (8) for sufficiently large n

$$\begin{aligned}\Phi(0)\, P[\varphi] - P(S_n^* \leqslant 0, \varphi) &\geqslant c_3 \frac{1}{\sqrt{n}} \sum_{v=v_0}^{n^{1/2\delta}} \frac{1}{v^{\tau_2+1-\delta}} \\ &\geqslant c_4 \frac{1}{\sqrt{n}} (n^{1/2\delta})^{\delta-\tau_2} \underset{(4)}{=} c_4 n^{-\tau_1}\end{aligned}$$

i.e., (ii) is fulfilled.

Proof of Example 8. Let $a = c(P \circ X_1)$, where $c(P \circ X_1)$ is the constant occurring in Lemma 3. Let $\varphi = \varphi_{\alpha,\beta} := \sum_{v \in \mathbb{N}} \varphi_v$ where $\varphi_v = (1/v^{1+\alpha})(\lg v)^\beta 1_{\{S_v^* \geqslant a\}}$. Then $\varphi \in \mathscr{L}_r$ and

$$\begin{aligned}d_r(\varphi, \mathscr{A}_n) &\leqslant \left\| \sum_{v>n} \varphi_v \right\|_r \leqslant \sum_{v>n} \|\varphi_v\|_r \\ &\leqslant \sum_{v>n} \frac{1}{v^{1+\alpha}} (\lg v)^\beta = O(n^{-\alpha}(\lg n)^\beta). \qquad (1)\end{aligned}$$

Hence (i) is fulfilled.

Applying Lemma 3 to $v \leqslant n/2 \wedge n/a^2$ and $B = \{S_v^* \geqslant a\} \in \mathscr{A}_v$, we obtain

$$\begin{aligned}&\Phi(0)\, P[\varphi_v] - P(S_n^* \leqslant 0, \varphi_v) \\ &\quad = \frac{1}{v^{1+\alpha}} (\lg v)^\beta \,(\Phi(0)\, P(B) - P(S_n^* \leqslant 0, B)) \\ &\quad \geqslant c_1 \frac{1}{v^{1+\alpha}} (\lg v)^\beta \sqrt{\frac{v}{n}}\, aP\{S_v^* \geqslant a\}.\end{aligned}$$

Hence there exists $c_2 = c_2(P \circ X_1)$ and $v_0 = v_0(P \circ X_1) \in \mathbb{N}$ such that

$$\Phi(0)\, P[\varphi_v] - P(S_n^* \leqslant 0, \varphi_v) \geqslant c_2 \frac{1}{\sqrt{n}} \frac{1}{v^{1/2+\alpha}} (\lg v)^\beta$$

if $v_0 \leqslant v \leqslant [n/2 \wedge n/a^2] =: j(n)$. This implies for sufficiently large n

$$\sum_{\nu=\nu_0}^{j(n)} (\Phi(0)\, P[\varphi_\nu] - P(S_n^* \leqslant 0, \varphi_\nu))$$

$$\begin{aligned} &\geqslant c_3 n^{-1/2} \lg\lg n; && \alpha = \tfrac{1}{2}, \beta = -1 \\ &\geqslant c_3 n^{-1/2} (\lg n)^{\beta+1}; && \alpha = \tfrac{1}{2}, \beta > -1 \\ &\geqslant c_3 n^{-\alpha} (\lg n)^{\beta}; && 0 < \alpha < \tfrac{1}{2}. \end{aligned} \tag{2}$$

As $P \circ X_1 = P \circ (-X_1)$ and $P \circ X_1$ is nonatomic we have by Lemma 8 $P(S_n^* \leqslant 0, S_\nu^* \geqslant a) \leqslant \frac{1}{2} P(S_\nu^* \geqslant a)$ and therefore

$$\Phi(0)\, P[\varphi_\nu] - P(S_n^* \leqslant 0, \varphi_\nu) \geqslant 0 \quad \text{for all } \nu, n \in \mathbb{N}. \tag{3}$$

Hence (2) and (3) directly imply (ii).

Proof of Example 9. Let $X_n, n \in \mathbb{N}$, be i.i.d. such that $P \circ X_1$ has density $p(t) = (c_1/|t|^{s+1} [\lg |t|]^2)\, 1_{[2,\infty)}(|t|)$ with respect to the Lebesgue measure. Then $X_n \in \mathscr{L}_s(\mathbb{R})$ and $P[X_n] = 0, n \in \mathbb{N}$. Let $g(t) = t^{s/r} 1_{[2,\infty)}(t)$ and put $\varphi = g \circ X_1$. Then $0 \leqslant \varphi \in \mathscr{L}_r(\mathbb{R})$ and $d_r(\varphi, \mathscr{A}_n) = 0, n \in \mathbb{N}$. Put $\tau_1 := \frac{1}{2} \cdot (s - s/r)$, then $0 < \tau_1 < \frac{1}{2}$. Hence it suffices to prove

$$\Phi(0)\, P[\varphi] - P(S_n^* \leqslant 0, \varphi) \geqslant c \frac{n^{-\tau_1}}{(\lg n)^2} \quad \text{for sufficiently large } n. \tag{1}$$

Using the Theorem of Berry–Esseen and Lemma 1, we have for sufficiently large n

$$\begin{aligned} &\Phi(0)\, P[\varphi] - P(S_n^* \leqslant 0, \varphi) \\ &\quad = \int \Phi(0)\, g(X_1)\, dP - \int g(X_1)\, P(S_n^* \leqslant 0 \mid \mathscr{A}_1)\, dP \\ &\quad = \int \Phi(0)\, g(X_1)\, dP - \int g(X_1)\, F_{n-1}\left(-\frac{1}{\sigma \sqrt{n-1}} X_1 \right) dP \\ &\quad \geqslant \int_2^\infty \left[\Phi(0) - \Phi\left(-\frac{1}{\sigma\sqrt{n-1}}\, t \right) \right] g(t) (P \circ X_1)(dt) - \frac{c_2}{\sqrt{n}} \\ &\quad \geqslant c_1 \int_2^\infty \left[\Phi(0) - \Phi\left(-\frac{1}{\sigma\sqrt{n-1}}\, t \right) \right] \frac{t^{s/r-(s+1)}}{[\lg t]^2}\, dt - \frac{c_2}{\sqrt{n}} \\ &\quad \geqslant c_3 (n-1)^{1/2 + 1/2 \cdot [s/r - (s+1)]} \int_2^\infty \frac{u^{s/r-(s+1)}}{[\lg |u| \sqrt{n-1}\,]^2}\, du - \frac{c_2}{\sqrt{n}} \\ &\quad \geqslant c_4 n^{-\tau_1} \int_2^3 \frac{u^{s/r-(s+1)}}{[\lg(3\sqrt{n-1})]^2}\, du - \frac{c_2}{\sqrt{n}} \geqslant c \frac{n^{-\tau_1}}{(\lg n)^2}, \end{aligned}$$

i.e., (1) is proved.

5. Auxiliary Lemmata

In this section we collect all lemmata which are needed for the proofs of the results and examples of Sections 3 and 4.

1. Lemma. *Let $X_n \in \mathscr{L}_3(\mathbb{R}^k)$, $n \in \mathbb{N}$, be i.i.d. with positive definite covariance matrix. Then we have for $x \in \mathbb{R}^k$ and $v, n \in \mathbb{N}$ with $v < n$ that*

$$\omega \to F_{n-v}\left(\sqrt{\frac{n}{n-v}}\,x - \sqrt{\frac{v}{n-v}}\,S_v^*(\omega)\right)$$

is a version of $P(S_n^* \leqslant x \mid \mathscr{A}_v)$.

Proof. Direct computation.

2. Lemma. *Let $X_n \in \mathscr{L}_3(\mathbb{R}^k)$, $n \in \mathbb{N}$, be i.i.d. with covariance matrix I. Let $f\colon \mathbb{R}^k \to [-1, 1]$ be a Berry–Esseen function. Then there exists a constant $c = c(k)$ such that for $v < n$*

$$|P(f \circ S_n^* \mid \mathscr{A}_v) - \Phi_{0,I}[f]| \leqslant \frac{c_f}{\sqrt{n-v}} + c\left[\frac{v}{n} + \sqrt{\frac{v}{n-v}}\,|S_v^*|\right].$$

Proof. According to Lemma 1 we have that for $v < n$

$$\omega \to F_{n-v}\left(\sqrt{\frac{n}{n-v}}\,x - \sqrt{\frac{v}{n-v}}\,S_v^*(\omega)\right)$$

is a version of $P(S_n^* \leqslant x \mid \mathscr{A}_v)$. Therefore

$$P(f \circ S_n^* \mid \mathscr{A}_v) = \int f(x)\, F_{n-v}\left(\sqrt{\frac{n}{n-v}}\,dx - \sqrt{\frac{v}{n-v}}\,S_v^*\right).$$

Hence we obtain

$$\begin{aligned}
&|P(f \circ S_n^* \mid \mathscr{A}_v) - \Phi_{0,I}[f]| \\
&\quad\leqslant \left|\int f(x)\left(F_{n-v}\left(\sqrt{\frac{n}{n-v}}\,dx - \sqrt{\frac{v}{n-v}}\,S_v^*\right)\right.\right. \\
&\qquad\left.\left. - \Phi_{0,I}\left(\sqrt{\frac{n}{n-v}}\,dx - \sqrt{\frac{v}{n-v}}\,S_v^*\right)\right)\right| \\
&\qquad + \left|\int f(x)\left(\Phi_{0,I}\left(\sqrt{\frac{n}{n-v}}\,dx - \sqrt{\frac{v}{n-v}}\,S_v^*\right) - \Phi_{0,I}(dx)\right)\right| \\
&\quad= \left|\int f\left(\sqrt{\frac{n-v}{n}}\,x + \sqrt{\frac{v}{n}}\,S_v^*\right)(F_{n-v} - \Phi_{0,I})\,dx\right| \\
&\qquad + \left|\int \left[f\left(\sqrt{\frac{n-v}{n}}\,x + \sqrt{\frac{v}{n}}\,S_v^*\right) - f(x)\right]\Phi_{0,I}(dx)\right|.
\end{aligned}$$

Since f is a Berry–Esseen function Lemma 4 implies

$$\leqslant \frac{c_f}{\sqrt{n-\nu}} + c\left[1-\sqrt{\frac{n-\nu}{n}}+\sqrt{\frac{\nu}{n-\nu}}\,|S_\nu^*|\right],$$

i.e., the assertion.

3. LEMMA. *Let $X_n \in \mathscr{L}_3(\mathbb{R})$, $n \in \mathbb{N}$, be i.i.d. with positive variance. Then there exist a universal constant c and a constant $c(P \circ X_1)$ such that*

$$\Phi(0)\,P(B) - P(S_n^* \leqslant 0, B) \geqslant c\,\sqrt{k/n}\,aP(B)$$

if $a \geqslant c(P \circ X_1)$, $B \in \mathscr{A}_k$ with $B \subset \{S_k^* \geqslant a\}$ and $ka^2 \leqslant n$, $1 \leqslant k \leqslant n/2$.

Proof. The proof runs similar to the proof of Lemma 4 in [4].

4. LEMMA. *There exists a constant $c = c(k)$ such that for each measurable function $f\colon \mathbb{R}^k \to [-1, +1]$*

$$\left|\int (f(ax+b) - f(x))\,\Phi_{0,I}(dx)\right| \leqslant c\left[(1-a) + \frac{|b|}{a}\right]$$

for $0 < a \leqslant 1$, $b \in \mathbb{R}^k$.

Proof. It suffices to show that

$$\left|\int (f(ax) - f(x))\,\Phi_{0,I}(dx)\right| \leqslant c(1-a) \qquad \text{for} \quad 0 < a \leqslant 1 \tag{1}$$

$$\left|\int (f(x+b) - f(x))\,\Phi_{0,I}(dx)\right| \leqslant c\,|b| \qquad \text{for} \quad b \in \mathbb{R}^k. \tag{2}$$

Ad (1). W.l.g. $a \geqslant \frac{1}{2}$ (choose $c \geqslant 4$). We have

$$\int f(ax)\,\Phi_{0,I}(dx) = \frac{1}{a^k}\int f(y)\,\varphi_{0,I}\left(\frac{1}{a}\,y\right)dy$$

and hence

$$\left|\int (f(ax) - f(x))\,\Phi_{0,I}(dx)\right| \leqslant \int \left|\frac{1}{a^k}\,\varphi_{0,I}\left(\frac{1}{a}\,y\right) - \varphi_{0,I}(y)\right| dy.$$

Therefore it suffices to find constants c_1, c_2 such that for $\frac{1}{2} \leqslant a \leqslant 1$, $y \in \mathbb{R}^k$

$$\left|\frac{1}{a^k}\,\varphi_{0,I}\left(\frac{1}{a}\,y\right) - \varphi_{0,I}(y)\right| \leqslant (1-a)[c_1 + c_2|y|^2]\,\varphi_{0,I}(y). \tag{3}$$

Let $y \in \mathbb{R}^k$ be fixed and put

$$g(a) = \frac{1}{a^k} \varphi_{0,I}\left(\frac{1}{a} y\right) - \varphi_{0,I}(y) \qquad \text{for} \quad \tfrac{1}{2} \leqslant a \leqslant 1.$$

As $g(1) = 0$, we obtain from the mean value theorem

$$|g(a)| \leqslant (1-a) \sup_{1/2 \leqslant \xi \leqslant 1} |g'(\xi)|. \tag{4}$$

Furthermore

$$\begin{aligned} g'(\xi) &= -\frac{k}{\xi^{k+1}} \varphi_{0,I}\left(\frac{1}{\xi} y\right) + \frac{1}{\xi^k} \left\langle \varphi'_{0,I}\left(\frac{1}{\xi} y\right), -\frac{1}{\xi^2} y \right\rangle \\ &= -\frac{k}{\xi^{k+1}} \varphi_{0,I}\left(\frac{1}{\xi} y\right) + \frac{1}{\xi^{k+3}} \varphi_{0,I}\left(\frac{1}{\xi} y\right) |y|^2. \end{aligned} \tag{5}$$

Now (4) and (5) imply (3).

Ad (2). Let w.l.g. $|b| \leqslant 1$. We have

$$\begin{aligned} \left| \int [f(x+b) - f(x)]\, \Phi_{0,I}(dx) \right| &= \left| \int f(x) [\varphi_{0,I}(x-b) - \varphi_{0,I}(x)]\, dx \right| \\ &\leqslant \int |\varphi_{0,I}(x-b) - \varphi_{0,I}(x)|\, dx. \end{aligned} \tag{6}$$

Using the mean value theorem and $e^{-(1/2)|z|^2} \leqslant e^{-(1/2)(|x|-1)^2}$, for $|x| > 1$ and $z \in [x-b, x]$, we obtain

$$\begin{aligned} |\varphi_{0,I}(x-b) - \varphi_{0,I}(x)| &\leqslant |b| \sup_{z \in [x-b,x]} |\varphi'_{0,I}(z)| \\ &= |b| \sup_{z \in [x-b,x]} |z|\, \varphi_{0,I}(z) \\ &\leqslant |b|(|x|+1) \sup_{z \in [x-b,x]} \varphi_{0,I}(z) \\ &\leqslant |b|(|x|+1)\{1_E(x) + e^{-(1/2)(|x|-1)^2}\} \end{aligned} \tag{7}$$

where $E = \{z \in \mathbb{R}^k : |z| \leqslant 1\}$. Now (6) and (7) imply (2).

5. Lemma. *Let $1 < r < \infty$ and $\varphi \in \mathscr{L}_r(\mathbb{R})$. Let $\mathscr{A}_0 \subset \mathscr{A}$ be a sub-σ-field of $\mathscr{A}$ and φ_0 an $\mathscr{A}_0$-measurable function with*

$$\|\varphi - \varphi_0\|_1 = d_1(\varphi, \mathscr{A}_0).$$

Then

$$\|\varphi_0\|_r \leqslant 2\|\varphi\|_r.$$

Proof. Let $Q: \Omega \times \mathscr{A}_0 \to [0, 1]$ be a regular conditional distribution of φ given $\mathscr{A}_0$. It is well known that $\varphi_0(\omega)$ is for P-a.a. $\omega \in \Omega$ a median of the p-measure $Q(\cdot, \omega) | \mathscr{B}$ (see [5]). Hence

$$|\varphi_0(\omega)| \leqslant 2 \int |x| \, Q(dx, \omega) \qquad P\text{-a.e.}$$

Then the convexity inequality implies

$$|\varphi_0(\omega)|^r \leqslant 2^r \int |x|^r \, Q(dx, \omega) \qquad P\text{-a.e.} \tag{1}$$

As $\int (\int |x|^r \, Q(dx, \omega)) \, P(d\omega) = \int |\varphi(\omega)|^r \, P(d\omega)$, integration of (1) yields the assertion.

6. LEMMA. *Let $s \geqslant 3$ and $X_n \in \mathscr{L}_s(\mathbb{R}^k)$, $n \in \mathbb{N}$, be i.i.d. with $P(X_1) = 0$ and covariance matrix I. Then there exists a constant $c = c(s, k)$ such that*

$$P\{|S_n^*| \geqslant t\} \leqslant c \frac{\rho_s}{t^s n^{(s-2)/2}} \qquad \text{for all } t > 0 \text{ with } t^2 \geqslant (s-1) \lg n.$$

Proof. Apply Theorem 17.11 of [1] to i.i.d. random variables with Cov $X_j = I$ and $\delta = 1$.

7. LEMMA. *Let $s \geqslant 2$ and let $X_n \in \mathscr{L}_s(\mathbb{R}^k)$, $n \in \mathbb{N}$, be i.i.d. with $P[X_1] = 0$ and covariance matrix I. Then there exists a constant $c = c(s, k)$ such that*

$$\|S_n^*\|_s \leqslant c \rho_s^{1/s}.$$

Proof. For $k = 1$ use Theorem 2 of [2, p. 356] and apply the proof of Corollary 2 of [2, p. 357]. The case $k > 1$ follows directly from the case $k = 1$.

8. LEMMA. *Let $X_n \in \mathscr{L}_3(\mathbb{R})$ be i.i.d. with positive variance such that $P \circ X_1 = P \circ (-X_1)$ and $P \circ X_1$ is nonatomic. Then we have for all $a > 0$ and $r, n \in \mathbb{N}$*

$$P(S_n^* \leqslant 0, S_r^* \geqslant a) \leqslant \tfrac{1}{2} P(S_r^* \geqslant a).$$

Proof. It suffices to show

$$P(S_n^* \leqslant 0, S_r^* \geqslant a) \leqslant P(S_n^* > 0, S_r^* \geqslant a).$$

The case $r = n$ is trivial. The cases $r < n$ and $r > n$ follow by using Lemma 1.

References

1. R. N. Bhattacharya and R. R. Rao, "Normal Approximation and Asymptotic Expansions," Wiley, New York, 1976.
2. Y. S. Chow and H. Teicher, "Probability Theory," Springer-Verlag, New York, 1978.
3. D. Landers and L. Rogge, Inequalities for conditioned normal approximations, *Ann. of Probab.* **5** (1977), 595–600.
4. D. Landers and L. Rogge, Exact approximation orders in the conditional central limit theorem, *Z. Wahrsch. Verw. Gebiete* **66** (1984), 227–244.
5. T. Shintani and T. Ando, Best approximants in L_1-spaces, *Z. Wahrsch. Verw. Gebiete* **33** (1975), 33–39.